Satellite Altimeter

We shall take earth radii as our distance unit, 1, and x as the satellite distance above the surface of the earth (altitude). See Figure 1 below.

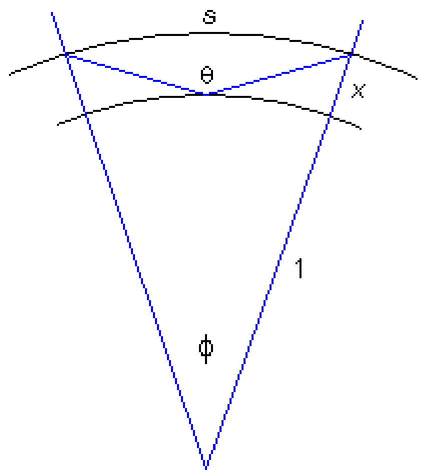

Figure 1.

Gravitational force $=$ centrifugal force

$$\frac{GMm}{(1+x)^2} = m\frac{v^2}{(1+x)}$$

Simplify

$$\frac{GM}{(1+x)} = v^2 \qquad (1)$$

Since

$$\frac{GM}{1^2} = g$$

then (1) becomes

$$\frac{g}{(1+x)} = v^2 \qquad (2)$$

Circumference of the circular orbit $=$ velocity times period

$$2\pi(1+x) = vT$$

Replace v using (2)

$$2\pi(1+x) = T\sqrt{\frac{g}{(1+x)}}$$

Rearrange to get a form of Kepler's third law

$$(1+x)^{3/2} = \frac{T\sqrt{g}}{2\pi} \qquad (3)$$

A satellite arc, s, subtended by an angle, θ, as measured by on observer on the surface of the earth would be different than the angle, ϕ, between the position vectors from the center of the earth. One reason for this difference is that the observer to satellite distance is shorter than the center of the earth to satellite distance. Lets assume that the satellite passes directly overhead, through the zenith, and we measure the time, t, between two positions at equal angles from the zenith. We want to find a relationship to give us the distance, x, when we measure the time, t, it takes for a satellite to pass through the arc θ.

ϕ is radians so

$$\phi = \frac{s}{1+x} \text{ and } \phi = \omega t$$

Since

$$\omega = \frac{2\pi}{T}$$

then

$$\frac{s}{1+x} = 2\pi\frac{t}{T}$$

$$\frac{s}{2\pi t} = \frac{(1+x)}{T} = \frac{\sqrt{g}(1+x)}{2\pi\ (1+x)^{3/2}} \qquad \text{using } (3)$$

Finally

$$\frac{s}{t} = \sqrt{\frac{g}{(1+x)}} \qquad (4)$$

Now we use some geometry to replace s by θ. Since $s \approx$ a straight line

$$\frac{s/2}{x} \approx \tan(\theta/2)$$

$$s = 2x\tan(\theta/2) \tag{5}$$

Substitute (5) into (4) to get

$$\frac{2x\tan(\theta/2)}{t} = \sqrt{\frac{g}{(1+x)}}$$

Solving for x

$$x = \frac{1}{2}\frac{t}{\tan(\theta/2)}\frac{\sqrt{g}}{\sqrt{(1+x)}} \tag{6}$$

If we take $g = 1.538 \times 10^{-6}\, er/s^2$, and let $\theta = 30°$, then

$$x\,@\,30° = 0.0023142\, er * \frac{t}{\sqrt{(1+x)}}$$

If we time satellite passage in seconds, t, through a $30°$ arc then this formula can be solved to find x, the satellites height above the surface. Some results are summarized in the table below

t	x	miles
10	.022	87
20	.045	178
30	.067	265
40	.089	353
50	.110	435
60	.131	519
70	.151	598
80	.171	677

The line $m = 8.42\,t + 10$ fits this data closely. This chart would be useful for making a crude determination of satellite altitude with a stopwatch, and is in fact the chart used by my Satellite Altimeter.

If the satellite is measured at two times not exactly straddling the zenith, then we must alter our formula. Any deviation from the exact straddle will result in x appearing to be larger than it really is. An angle, φ, related to the elevation now figures into formula (6) as

$$x = \frac{1}{2}\frac{t\sin^2\varphi}{\tan(\theta/2)}\frac{\sqrt{g}}{\sqrt{(1+x)}} \tag{7}$$

The term $\sin\varphi$ may be calculated from the sight angle unit vectors $\overline{\boldsymbol{ll}[0]}$ and $\overline{\boldsymbol{ll}[1]}$, and the observer position vector $\overline{\boldsymbol{rd}}$. See Figure 2 below.

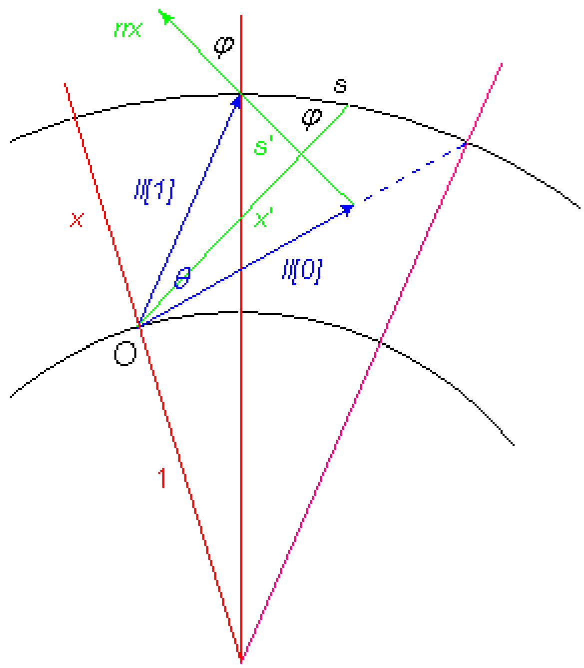

Figure 2.

Since $$\frac{s'/2}{s/2} = \sin\varphi \;\Rightarrow\; s = \frac{s'}{\sin\varphi}$$

let $$\overline{\boldsymbol{rrx}} = \left(\overline{\boldsymbol{ll}[1]} - \overline{\boldsymbol{ll}[0]}\right)$$

$$\overline{\boldsymbol{rry}} = \overline{\boldsymbol{rd}} \times \overline{\boldsymbol{rrx}}$$

then $$\sin\varphi = \frac{\|\overline{\boldsymbol{rry}}\|}{\|\overline{\boldsymbol{rd}}\|\,\|\overline{\boldsymbol{rrx}}\|}$$

Continuing $$\frac{s'/2}{x'} \approx \tan(\theta/2)$$

$$s' = 2x'\tan(\theta/2)$$

$$\frac{x}{x'} \approx \sin\varphi$$

so that
$$\frac{s}{t} = \frac{2x\tan(\theta/2)}{t\sin^2\varphi} = \sqrt{\frac{g}{(1+x)}}$$

Solving for x we get (7). This is the approach used in ELFIND to get an initial approximation to the length of the satellite position vector.